Signal Analysis & Communication ECE 355

Ch. 5.3: Properties of DTFT

Lecture 25

13-11-2023

Ch. 5.3 : PROPERTIES OF DTFT [Similar to the properties of CTFT 1. provides insights 2. reduces complexity]

Recall:

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

① Periodic:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

② Linearity:

$$a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{\mathcal{F}} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

③ Time & Frequency Shifting

$$x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega}) \quad — \text{(A)}$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)}) \quad — \text{(B)}$$

Proof of (B)

$$\text{RHS} \xleftrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi}\int_{2\pi} X(e^{j(\omega-\omega_0)}) e^{j\omega n} d\omega$$

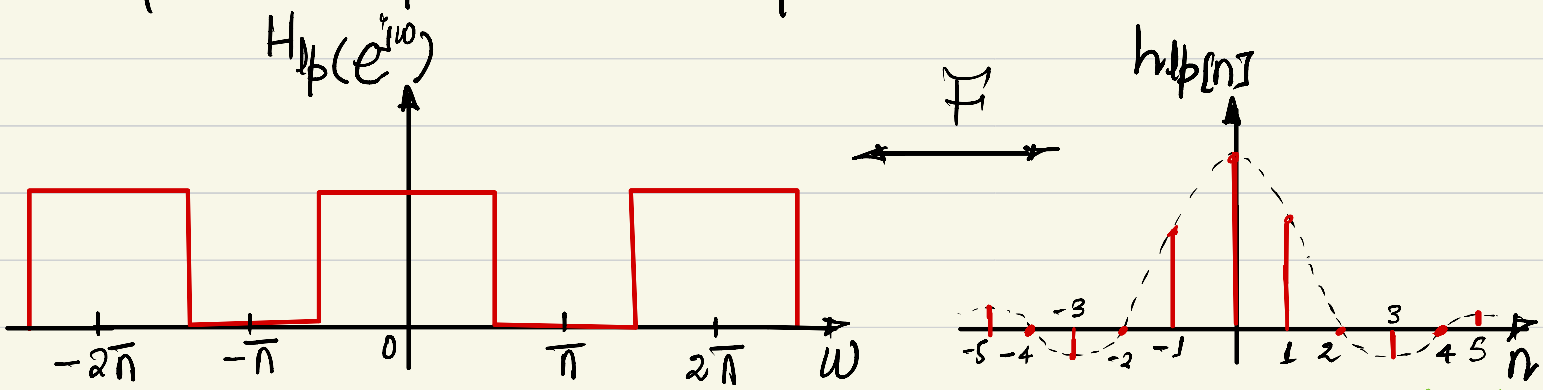
$$\overset{\alpha=\omega-\omega_0}{=\!=} \underbrace{\frac{1}{2\pi}\int_{2\pi} X(e^{j\alpha}) e^{j\alpha n}}\, \underbrace{e^{j\omega_0 n}}\, d\alpha$$

$$= x[n] e^{j\omega_0 n} = \text{LHS}$$

Example

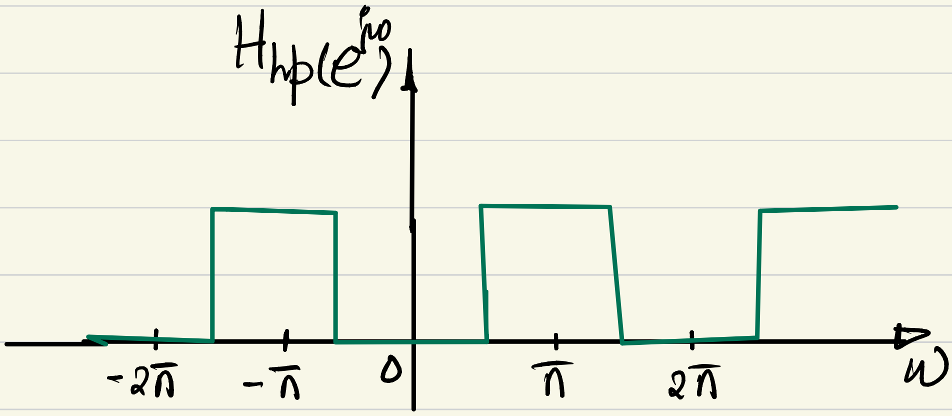
Ideal low-pass & high-pass filters.

- The freq. response of an ideal LPF & its corresponding impulse response is plotted for DT aperiodic as:



* zero at even values of 'n' (for this considered example)

- The freq. response of an ideal HPF is the delayed version of ideal LPF.

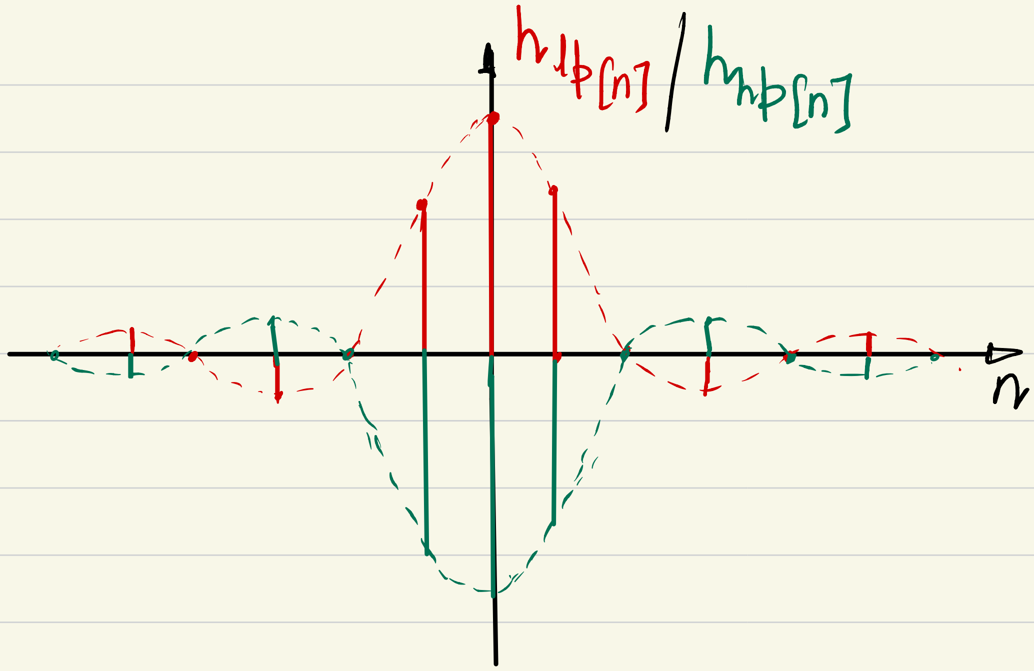


- That is:

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

- And the respective impulse response is given using freq. shifting property.

$$h_{hp}[n] = e^{j\pi n} h_{lp}[n] = (-1)^n h_{lp}[n]$$



④ Conjugation:

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

Cases

I. If $x[n]$ is Real

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

II. If $x[n]$ is Real & Even

$$x[n] \xleftrightarrow{\mathcal{F}} Re\{X(e^{j\omega})\}$$

III. If $x[n]$ is Real & Odd

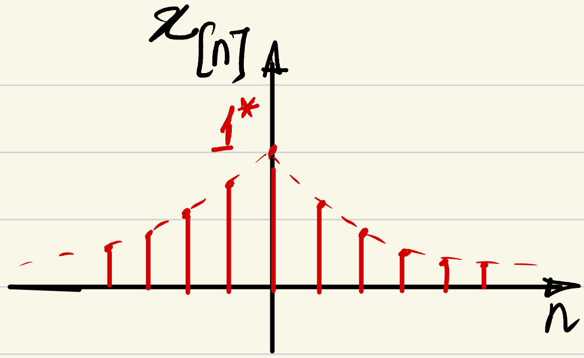
$$x[n] \xleftrightarrow{\mathcal{F}} Im\{X(e^{j\omega})\}$$

Example

$x[n] = a^{|n|}$; $|a| < 1$ — ⓔ Even Function

Consider

$$x_1[n] = a^n u[n]$$



$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

Eqn ⓒ can be expressed as:

$$x[n] = a^n u[n] + a^{-n} u[-n] - \underline{\delta[n]}\ *$$

$$x[n] = 2\,\text{Even}\{x_1[n]\} - \delta[n]$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} 2\,\text{Re}\left\{\frac{1}{1-ae^{-j\omega}}\right\} - 1$$

$$= \dots = \frac{1-a^2}{1+a^2-2a\cos\omega}$$

⑤ Differencing & Accumulation:

Differencing

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1-e^{-j\omega(1)})X(e^{j\omega})$$

(By Linearity)

$$x[n] - x[n-k] \overset{\mathcal{F}}{\longleftrightarrow} (1-e^{-j\omega k})X(e^{j\omega})$$

Accumulation

$$\sum_{m=-\infty}^{n} x_{[m]} \xleftrightarrow{\mathcal{F}} ?$$

We know (from Lecture 8) that

$$\sum_{m=-\infty}^{n} x_{[m]} = x_{[n]} * u_{[n]}$$

- As with CTFT, we will see later that Convolution in time-domain is multiplication in freq.-domain.

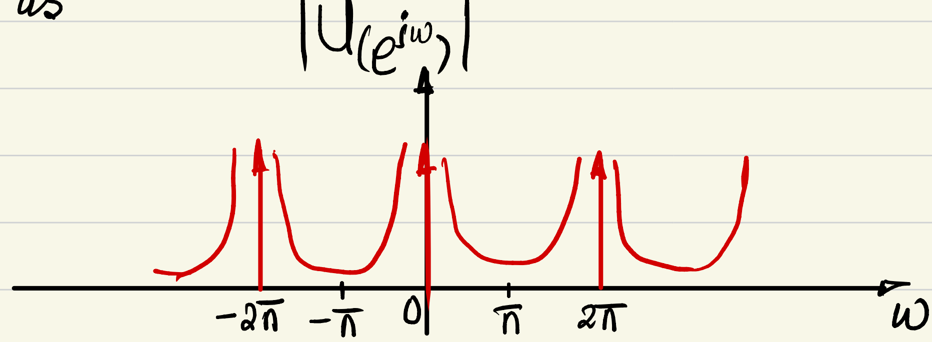
$$\xleftrightarrow{\mathcal{F}} X_{(e^{j\omega})} \cdot U_{(e^{j\omega})}$$

- To find $U_{(e^{j\omega})}$, recall: $u_{(t)} \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\,\delta_{(\omega)}$ (CTFT)

- We can show that

$$U_{(e^{j\omega})} = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\,\delta_{(\omega-2\pi k)}$$

- Plotted as



- Hence,

$$\sum_{m=-\infty}^{n} x_{[n]} \longleftrightarrow X_{(e^{j\omega})}\, U_{(e^{j\omega})}$$

Integral multiple of 2π results in zero!

$$= \frac{1}{1-e^{-j\omega}} X_{(e^{j\omega})} + \sum_{k=-\infty}^{\infty} \pi X_{(e^{j2\pi k})}\, \delta_{(\omega-2\pi k)}$$

$$= \frac{1}{1-e^{-j\omega}} X_{(e^{j\omega})} + \pi X_{(e^{j0})} \sum_{k=-\infty}^{\infty} \delta_{(\omega-2\pi k)}$$

exist at $\omega = 2\pi k$

⑥ Time Reversal:

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Proof

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]\, e^{-j\omega n}$$

$n=-m$

$$= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j(-\omega)m}$$

⑦ Time Scaling

- Recall for CT signals:

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \quad — \text{Ⓓ}$$

- Because of the discrete nature of the time index for DT signals, the relation b/w time & freq. scaling takes on somewhat different form.

- If we try to define the sig. $x[an]$, we run into difficulties if $a \notin \mathbb{Z}$
(as DT sig. is defined only at integer values)

- For $a \in \mathbb{Z}$

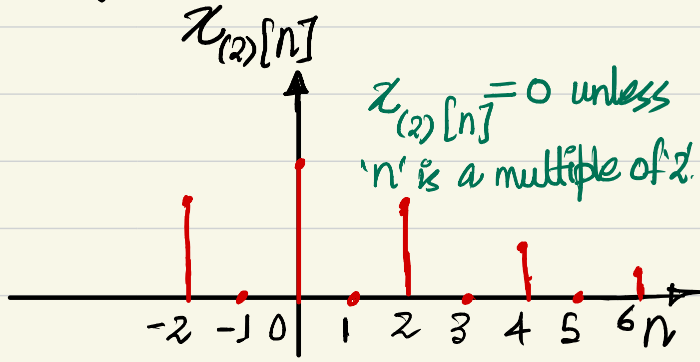
$|a| > 1$

- Its not speeding up (contraction) of the sig., since 'n' can take on only integer values.
- E.g., the signal $x_{[2n]}$ consists of the even samples of $x_{[n]}$ alone.

- Therefore, we define a sig. for getting the result that closely parallels eqn. (D) as:

$$x_{(k)}[n] \triangleq \begin{cases} x(n/k) & , \text{ only if 'n' is multiple of 'k'} \\ 0 & , \text{ otherwise} \end{cases}$$

- For example, for $k=2$, $x_{(k)}[n]$ is obtained from $x_{[n]}$ by placing '$k-1$' zeros b/w successive values of the original sig.



Slowing down (spreads out)

- Its FT is given as

$$X_k(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n}$$

$$= \sum_{r=-\infty}^{\infty} \underbrace{x_{(k)}[rk]}_{x_{[r]}} e^{-j\omega rk}$$

$x_{[r]}$ ← variable now is 'r'. ('k' fixed!)

* $x_{(k)}[n] = 0$ unless 'n' is a multiple of 'k'

$$n = rk$$

$$\therefore \quad X_k(e^{j\omega}) = X(e^{j\underline{\underline{k\omega}}})$$

Transform gets compressed!